PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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On the Self-Inductance of Circular Coils of Rectangular Section

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IX. *On the Self-inductance of Circular Coils of Rectangular Section.*

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As an approximate formula for the calculation of the self-inductance of a coil of rectangular section,

$$L = 4\pi a n^2 \left\{\log \frac{8a}{r} - 2\right\}$$

was first given by Maxwell,* where a is the mean radius and r the geometric mean distance of the section of the coil from itself, the current being supposed to be uniformly distributed over the section.

In the following paper it will be shown that the same formula will give the self-inductance to any order of accuracy when in it are substituted for a and r the mean radius and the G.M.D. respectively, each suitably modified by small quantities which depend on a and on the section of the coil, provided, of course, that the series for L is convergent.

Tables will be given by means of which the modified values of a and r for any coil of rectangular section can be found, and which, when substituted in the above formula, will give L correct to the fourth order, uniform current density over the section being assumed.

1. For the purpose of this paper the mutual inductance of two coaxal circles can best be obtained after Weinstein† by substituting in Maxwell's exact elliptic integral formula‡

$$M = 4\pi\sqrt{ab}\left\{\left(\frac{2}{k} - k\right) F - \frac{2}{k} E\right\},$$

the series expressions for F and E in terms of the complementary modulus k'.

Thus we obtain

$$M = 4\pi\sqrt{ab}\left\{\log\frac{4}{k'} - 2 + \frac{3}{4}k'^2\left(\log\frac{4}{k'} - 1\right) + \frac{1}{64}k'^4\left(33\log\frac{4}{k'} - \frac{81}{2}\right)\right\},$$

which is rapidly convergent when k', the ratio of the least to the greatest distance between the circles, is small.

* 'Elect. and Mag.,' vol. II., § 706.

† 'Wied. Ann.,' 21, p. 344, 1884.

‡ 'Elect. and Mag.,' vol. II., § 701.

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Let these two circles be filaments A and B in the rectangular conductor whose section is PQRS, fig. 1. Then, if x and y be the co-ordinates of B relative to axes through A parallel to the sides of the rectangle,

$$k'^2 = \frac{x^2+y^2}{x^2+(2a+y)^2},$$

and on substitution in the above we obtain, as ROSA and COHEN* have done,

$$\begin{aligned}M = 4\pi a\Big[\log\frac{8a}{r}\Big(1+\frac{y}{2a}+\frac{y^2+3x^2}{16a^2}-\frac{y^3+3yx^2}{32a^3}+\frac{17y^4+42y^2x^2-15x^4}{1024a^4}\Big)\\ -2-\frac{y}{2a}+\frac{3y^2-x^2}{16a^2}-\frac{y^3-6yx^2}{48a^3}-\frac{19y^4+534y^2x^2-93x^4}{6144a^4}\Big]\end{aligned}$$

$= M_0$ say,

where a is the radius of the circle A and $r^2 = x^2+y^2$.

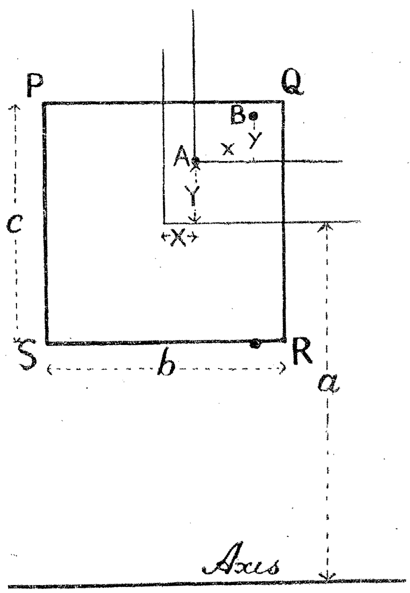

Fig. 1.

If the co-ordinates of the A circle, referred to axes through the centre of the rectangle, be X and Y, then a in the above expression for M becomes $a+Y$ where a now and in what follows is the mean radius of the coil.

This substitution is most easily carried out by aid of TAYLOR'S Theorem. Thus the complete expression for M is given by

$$M = M_0+Y\frac{dM_0}{da}+\frac{Y^2}{1\,.\,2}\frac{d^2M_0}{da^2}+\&c.$$

* 'Bull. Bureau Standards,' vol. 2, p. 364, 1906.

2. It is well known that the self-inductance of a single circular conductor with rectangular section for uniform current density is given by

$$\frac{1}{b^2c^2}\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\int_{-\frac{1}{2}c-\mathrm{Y}}^{\frac{1}{2}c-\mathrm{Y}}\int_{-\frac{1}{2}b-\mathrm{X}}^{\frac{1}{2}b-\mathrm{X}} \mathrm{M}\,dx\,dy\,d\mathrm{X}\,d\mathrm{Y},$$

and for a coil of n turns is n^2 times this if MAXWELL'S correction* for space between the wires be neglected, b being the breadth and c the radial depth of the rectangle.

The evaluation of this definite integral, even for the second order terms, has presented considerable difficulties. The first correct result to this order was that given by WEINSTEIN.† So far as I am aware no one has published a determination of it to the fourth order. By a method indicated in the appendix to this paper the integration can be carried out to the fourth order without difficulty and to still higher orders if desired.

Thus the following expression for L has been obtained

$$\begin{aligned}\mathrm{L} = 4\pi a n^2\Bigg[\log\frac{8a}{d}+\frac{1}{12}+\frac{u+v}{12}-\frac{2}{3}(w+w')\\
+\frac{1}{2^5\,.\,3\,.\,a^2}\left\{(3b^2+c^2)\log\frac{8a}{d}+\frac{1}{2}b^2u-\frac{1}{10}c^2v-\frac{16}{5}b^2w+\frac{69}{20}b^2+\frac{221}{60}c^2\right\}\\
+\frac{1}{2^{11}\,.\,3\,.\,5\,.\,a^4}\Bigg\{\left(-30b^4+35b^2c^2+\frac{22}{3}c^4\right)\log\frac{8a}{d}-\frac{115b^4-480b^2c^2}{12}u\\
-\frac{23}{28}c^4v+\frac{6b^4-7b^2c^2}{21}\,.\,2^8\,.\,w\\
-\frac{36590b^4-2035b^2c^2-11442c^4}{2^3\,.\,3\,.\,5\,.\,7}\Bigg\}\Bigg],\end{aligned}$$

in which

a = mean radius, b = breadth, c = depth, $d=\sqrt{b^2+c^2}$ = diagonal of the rectangle,

and

$$u=\frac{b^2}{c^2}\log\frac{d^2}{b^2},\qquad v=\frac{c^2}{b^2}\log\frac{d^2}{c^2},$$

$$w=\frac{b}{c}\tan^{-1}\frac{c}{b},\qquad w'=\frac{c}{b}\tan^{-1}\frac{b}{c}.$$

3. If r be the G.M.D. of the rectangle from itself, it is well known that

$$\log r = \log d-\phi$$

where

$$\phi=\frac{u+v+25}{12}-\frac{2}{3}(w+w').$$

* 'Elect. and Mag.,' vol. II., § 693. ROSA (see 'Bull. Bureau Standards,' vol. 3, p. 37, 1907) has greatly improved on MAXWELL'S correction.

† 'WIED. Ann.,' 21, p. 329, 1884.

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Substituting for d in terms of r and ϕ in the above expression for L we obtain

$$L = 4\pi a n^2 \left[\log\frac{8a}{r} - 2 + \frac{d^2}{a^2}\left\{p_2 \log\frac{8a}{r} + q_2\right\} + \frac{d^4}{a^4}\left\{p_4 \log\frac{8a}{r} + q_4\right\}\right],$$

where

$$p_2 = \frac{1}{2^5 . 3}\,\frac{3b^2+c^2}{d^2},$$

$$p_4 = \frac{1}{2^{11} . 3^2 . 5}\,\frac{-90b^4+105b^2c^2+22c^4}{d^4},$$

$$q_2 = \frac{1}{2^5 . 3 . d^2}\left\{\frac{1}{2}\,b^2u - \frac{1}{10}\,c^2v - \frac{16}{5}\,b^2w - (3b^2+c^2)\,\phi + \frac{69}{20}b^2 + \frac{221}{60}\,c^2\right\},$$

$$q_4 = \frac{1}{2^{11}.3^2.5.d^4}\left\{(90b^4-105b^2c^2-22c^4)\,\phi - \frac{69}{28}c^4v - \frac{115b^4-480b^2c^2}{4}\,u + \frac{6b^4-7b^2c^2}{7}\,.2^8.w - \frac{36590b^4-2035b^2c^2-11442c^4}{2^3 . 5 . 7}\right\}.$$

4. If

$$A = a\left(1 + m_1\frac{d^2}{a^2} + m_2\frac{d^4}{a^4}\right),$$

$$R = r\left(1 + n_1\frac{d^2}{a^2} + n_2\frac{d^4}{a^4} + n_3\frac{d^6}{a^6}\right),$$

m_1, m_2, n_1, n_2, n_3 can be determined so that

$$4\pi n^2 A\left(\log\frac{8A}{R} - 2\right)$$

shall differ very little from the value for L given in § 3.

After substituting for A and R in the above and expanding in a series in d^2/a^2, the first three terms of the expansion are identified with the corresponding terms of L in the usual way, and in addition, the coefficient of the fourth term of the expansion, that is the coefficient of d^6/a^6, is equated to zero. The n_3 term in R enables this to be done, with the result that a closer agreement is obtained between the proposed formula and that in § 3.

Thus

$$m_1 = p_2,\ n_1 = -(p_2+q_2),$$

$$m_2 = p_4,\ n_2 = -(p_4+q_4) + \tfrac{1}{2}(m_1-n_1)^2,$$

$$n_3 = (m_1-n_1)\left[m_2-n_2-\tfrac{1}{6}(m_1-n_1)(m_1+2n_1)\right].$$

Hence, when A and R have been so determined the formula

$$L = 4\pi\, A n^2\left(\log\frac{8A}{R} - 2\right),$$

will give the self-inductance of the coil correct to the fourth order.

It will be seen that in the application of this formula the coefficient n_3 need rarely be used. It becomes important, however, when the mean radius is less than the diagonal of the section and especially in this case when the section is square or nearly so.

It is obvious that in a similar way to the above, series for A and R could be obtained which, when substituted in the proposed formula would make it practically equivalent to L, no matter to what order the integration, if performed, had been carried out.

5. In order to render convenient the practical application of the above formula to the determination of self-inductances the following tables* have been prepared.

TABLE I.

G.M.D. $= r.$ $\quad d^2 = b^2+c^2.$

$\frac{c}{b}$ or $\frac{b}{c}$.	$\phi = \log_e \frac{d}{r}$.	$\frac{r}{b+c}$.
0·00	1·5	0·223130
0·025	1·474734	0·223328
0·05	1·451005	0·223455
0·10	1·407566	0·223599
0·15	1·368975	0·223664
0·20	1·334799	0·223686
0·25	1·304680	0·223686
0·30	1·278284	0·223675
0·35	1·255312	0·223658
0·40	1·235461	0·223639
0·45	1·218448	0·223619
0·50	1·203998	0·223601
0·55	1·191853	0·223584
0·60	1·181768	0·223570
0·65	1·173516	0·223558
0·70	1·166888	0·223548
0·75	1·161691	0·223540
0·80	1·157752	0·223534
0·85	1·154914	0·223530
0·90	1·153034	0·223527
0·95	1·151987	0·223525
1·00	1·151660	0·223525

Table I. contains (1) values of ϕ, that is of $\log_e \frac{d}{r}$, for different values of the ratio b/c or c/b (2) values of the ratio $r/b+c$ for different values of b/c. It will be noticed how nearly r the G.M.D. of a rectangle from itself is proportional to the sum of the

* All the tables given in this paper have been calculated with the greatest care by the aid of a "millionaire" calculating machine. Each separate series of numbers, not only the final series but every intermediate series that had to be determined, was calculated at least twice, the end terms and one or two intermediate terms of each series were carefully re-checked, and each series then examined by taking successive differences.

sides. These figures will enable the G.M.D. for values of c/b or b/c, intermediate to those given in the table, and consequently the first or important term of L for such intermediate values to be obtained with great accuracy.

Table II. contains the values of the coefficients m_1, m_2, n_1, n_2, n_3, for thick coils, that is for ones in which b is greater than c for different values of the ratio c/b, and Table III. contains the values of the same coefficients for thin coils, that is for ones in which b is less than c for different values of the ratio b/c.

TABLE II.

$\frac{c}{b}$.	$10^2 m_1$.	$10^4 m_2$	$10^2 n_1$.	$10^4 n_2$.	$10^6 n_3$.
0·00	3·12500	−9·766	0·78125	−8·647	− 6·9
0·025	3·12370	−9·746	0·69934	−8·179	− 8·2
0·05	3·11980	−9·688	0·61606	−7·663	− 9·7
0·10	3·10437	−9·461	0·44541	−6·505	−12·6
0·15	3·07916	−9·094	+0·26934	−5·202	−15·7
0·20	3·04487	−8·604	+0·08919	−3·795	−18·9
0·25	3·00245	−8·011	−0·09342	−2·326	−22·1
0·30	2·95298	−7·340	−0·27664	−0·838	−25·2
0·35	2·89764	−6·614	−0·45856	+0·630	−28·0
0·40	2·83764	−5·857	−0·63754	+2·045	−30·6
0·45	2·77417	−5·090	−0·81198	3·378	−32·8
0·50	2·70833	−4·332	−0·98060	4·610	−34·7
0·55	2·64166	−3·596	−1·14240	5·727	−36·1
0·60	2·57353	−2·895	−1·29662	6·725	−37·2
0·65	2·50622	−2·237	−1·44274	7·600	−37·8
0·70	2·43988	−1·626	−1·58048	8·356	−38·2
0·75	2·37500	−1·066	−1·70975	8·999	−38·2
0·80	2·31199	−0·556	−1·83060	9·536	−37·9
0·85	2·25115	−0·097	−1·94321	9·978	−37·5
0·90	2·19268	+0·314	−2·04787	10·333	−36·8
0·95	2·13672	+0·680	−2·14491	10·611	−35·9
1·00	2·08333	+1·004	−2·23473	10·824	−35·0
1·05	2·03255	+1·289	−2·31773	10·978	−33·9

It will have been noticed that the coefficient m_1 and m_2 are algebraic and can be easily calculated for any value of c/b. Those in the tables are given for convenience.

6. If the formula for L given in § 2 be written in the form

$$L = 4\pi a n^2 \left[\left(1 + m_1 \frac{d^2}{a^2} + m_2 \frac{d^4}{a^4}\right) \log \frac{8a}{d} - l_0 + l_1 \frac{d^2}{a^2} + l_2 \frac{d^4}{a^4}\right],$$

tables giving m_1, m_2, l_0, l_1, and l_2, for different values of c/b would also render easy the computation of the self inductances of coils. Such tables have been computed from WEINSTEIN'S formula by STEFAN,* but he is in error in thinking that the second order coefficient has the same value for a given value of b/c in a thin coil as it has for the

* 'WIED. Ann.,' 22, p. 113, 1884.

same value of c/b in a thick coil. The second order coefficients he gives are correct for thick coils.

In using the above formula with tables for the computation of L for values of c/b or b/c intermediate to those given in the tables, the value of l_0 which is part of the large or first order term will have to be obtained by interpolation, whereas in the

TABLE III.

$\frac{b}{c}$.	$10^2 m_1$.	$10^4 m_2$.	$10^2 n_1$.	$10^4 n_2$.	$10^6 n_3$.
0·00	1·04167	2·387	−3·21180	6·073	+ 0·5
0·025	1·04297	2·391	−3·23737	6·134	+ 0·6
0·05	1·04686	2·403	−3·25967	6·214	+ 0·5
0·10	1·06229	2·451	−3·29420	6·430	+ 0·1
0·15	1·08751	2·524	−3·31479	6·724	− 0·6
0·20	1·12179	2·614	−3·32107	7·090	− 1·7
0·25	1·16421	2·711	−3·31313	7·513	− 3·2
0·30	1·21368	2·806	−3·29150	7·978	− 5·1
0·35	1·26902	2·886	−3·25703	8·463	− 7·3
0·40	1·32902	2·943	−3·21091	8·949	− 9·8
0·45	1·39250	2·969	−3·15447	9·414	−12·4
0·50	1·45833	2·960	−3·08918	9·845	−15·0
0·55	1·52551	2·912	−3·01651	10·226	−17·7
0·60	1·59313	2·824	−2·93794	10·548	−20·3
0·65	1·66044	2·697	−2·85483	10·805	−22·8
0·70	1·72678	2·534	−2·76844	10·994	−25·2
0·75	1·79167	2·337	−2·67990	11·115	−27·4
0·80	1·85468	2·111	−2·59020	11·171	−29·3
0·85	1·91552	1·861	−2·50019	11·164	−31·1
0·90	1·97399	1·590	−2·41057	11·100	−32·6
0·95	2·02995	1·303	−2·32192	10·985	−33·9
1·00	2·08333	1·004	−2·23473	10·824	−35·0
1·05	2·13412	0·696	−2·14940	10·624	−36·2

method previously given, the whole of the first order term can be easily got with great accuracy, by making use of the nearly constant ratio of r to $b+c$ indicated by the figures given in the third column of Table I.

The coefficients l_0, l_1, l_2, occurred in the computation of Tables I., II., and III., m_1 and m_2 are the same as in these tables, and are in any case algebraic as

$$m_1 = \frac{1}{2^5 . 3} \frac{3b^2+c^2}{b^2+c^2},$$

$$m_2 = \frac{1}{2^{11} . 3^2 . 5} \frac{-90b^4+105b^2c^2+22c^4}{(b^2+c^2)^2}.$$

Table IV. gives the values of l_0, l_1, and l_2 for different values of the ratio c/b for thick coils, and of b/c for thin coils.

TABLE IV.

For both thick and thin coils.		For thick coils.			For thin coils.		
$\frac{c}{b}$ or $\frac{b}{c}$.	l_0.	$\frac{c}{b}$.	$10^2 l_1$.	$10^4 l_2$.	$\frac{b}{c}$.	$10^2 l_1$.	$10^4 l_2$.
0·00	0·500000	0·00	0·78125	6·510	0·00	3·73264	4·167
0·025	0·525266	0·025	0·78358	6·490	0·025	3·73250	4·161
0·05	0·548995	0·05	0·79098	6·427	0·05	3·73181	4·143
0·10	0·592434	0·10	0·81983	6·184	0·10	3·72716	4·058
0·15	0·631025	0·15	0·86679	5·794	0·15	3·71605	3·897
0·20	0·665201	0·20	0·93023	5·283	0·20	3·69664	3·655
0·25	0·695320	0·25	1·00821	4·677	0·25	3·66784	3·336
0·30	0·721716	0·30	1·09841	4·010	0·30	3·62925	2·951
0·35	0·744688	0·35	1·19836	3·313	0·35	3·58103	2·516
0·40	0·764539	0·40	1·30570	2·614	0·40	3·52385	2·050
0·45	0·781552	0·45	1·41799	1·940	0·45	3·45866	1·572
0·50	0·796002	0·50	1·53310	1·311	0·50	3·38668	1·099
0·55	0·808147	0·55	1·64911	0·740	0·55	3·30919	0·648
0·60	0·818232	0·60	1·76440	+0·238	0·60	3·22752	+0·231
0·65	0·826484	0·65	1·87761	−0·191	0·65	3·14294	−0·143
0·70	0·833112	0·70	1·98767	−0·546	0·70	3·05662	−0·468
0·75	0·838309	0·75	2·09376	−0·829	0·75	2·96960	−0·740
0·80	0·842248	0·80	2·19532	−1·044	0·80	2·88278	−0·959
0·85	0·845086	0·85	2·29194	−1·197	0·85	2·79693	−1·127
0·90	0·846966	0·90	2·38342	−1·294	0·90	2·71265	−1·245
0·95	0·848013	0·95	2·46966	−1·342	0·95	2·63045	−1·318
1·00	0·848340	1·00	2·55069	−1·349	1·00	2·55069	−1·349
1·05	0·848044	1·05	2·62659	−1·320	1·05	2·47369	−1·344

7. The only available means of testing the above methods of computing self-inductances and of finding the limit outside which they are practically reliable is to compare the results they give with those given by LORENZ's* exact elliptic integral formula for the self-inductance of a current sheet solenoid, which is

$$L = \frac{32}{3}\frac{\pi a^3}{d^2}\left\{\frac{2k^2-1}{k^3}E + \frac{1-k^2}{k^3}F - 1\right\},$$

where a is the radius, d the length of the solenoid, and

$$k^2 = \frac{4a^2}{4a^2+d^2}.$$

Thus consider the case of a solenoid whose length is twice its radius.

Here

$$\frac{d}{a} = 2, \qquad \frac{c}{b} = 0,$$

* 'WIED. Ann.,' 7, p. 161, 1879.

and from Table II.

$$A = (1+4\times 0{\cdot}03125-16\times 0{\cdot}0009766)\,a,$$

$$= 1{\cdot}109375a.$$

$$R = (1+4\times 0{\cdot}0078125-16\times 0{\cdot}0008647-64\times 0{\cdot}0000069)\,r,$$

$$= 1{\cdot}016973r.$$

so

$$\log_e \frac{8A}{R} = \log_e \frac{8a}{d}+\phi+\log_e \frac{1{\cdot}109375}{1{\cdot}016973},$$

(where ϕ is given in Table I.)

$$= \log_e 4+\phi+0{\cdot}086965,$$

$$= 2{\cdot}973259,$$

and

$$4\pi A\left(\log \frac{8A}{R}-2\right) = 4\pi a\times 1{\cdot}07970.$$

LORENZ's exact formula gives

$$L = 4\pi a\times 1{\cdot}08137.$$

Thus the error in this case is 1 part in 650.

When the comparison is made in less extreme cases we find the agreement with the Lorenz formula very close.

Thus when the length of the solenoid is equal to its radius (a) either of the methods of this paper give

$$L = 20{\cdot}7453a,$$

while LORENZ's formula gives

$$L = 20{\cdot}7463a,$$

showing an error of 1 part in 20,000, and when the length of the solenoid is half the radius we obtain

$$L = 28{\cdot}85332a,$$

as against the Lorenz value

$$L = 28{\cdot}85335a,$$

showing an error of only about 1 part in 1,000,000.

APPENDIX I.

In order to determine L to the fourth order we have seen that it is necessary to evaluate the definite integral

$$\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\int_{-\frac{1}{2}c-\mathrm{Y}}^{\frac{1}{2}c-\mathrm{Y}}\int_{-\frac{1}{2}b-\mathrm{X}}^{\frac{1}{2}b-\mathrm{X}} \mathrm{M}\,dx\,dy\,d\mathrm{X}\,d\mathrm{Y}$$

where

$$\mathrm{M} = \mathrm{P}+\mathrm{QY}+\mathrm{RY}^2+\mathrm{SY}^3+\mathrm{TY}^4,$$

P, Q, R, S, and T being functions of x and y.

If we proceed in the ordinary way by putting in the limits after each integration the expression becomes very cumbrous on account of the nature of some of the functions (log and $\tan^{-1}$) with which we have to deal.

By the method to be explained below all the integration will be carried out first and the limits introduced in an easy and symmetrical way at the finish.

1. Dealing first with P, the term independent of Y, if

$$\iint \mathrm{P}\,dx\,dy = \theta(xy),$$

the result, with limits introduced, of the integrations with respect to x and y will be

$$\theta(x_1y_1)-\theta(x_1y_2)-\theta(x_2y_1)+\theta(x_2y_2),$$

where

$$x_1 = \tfrac{1}{2}b-\mathrm{X}, \qquad x_2 = -\tfrac{1}{2}b-\mathrm{X},$$
$$y_1 = \tfrac{1}{2}c-\mathrm{Y}, \qquad y_2 = -\tfrac{1}{2}c-\mathrm{Y}.$$

We have now to evaluate four definite integrals of which the first is

$$\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \theta(x_1y_1)\,d\mathrm{X}\,d\mathrm{Y}.$$

Changing the variables to x_1 and y_1 and the limits accordingly, this integral is equal to

$$\int_c^0\int_b^0 \theta(x_1y_1)\,dx_1\,dy_1$$
$$= \phi(0,0)-\phi(0,c)-\phi(b,0)+\phi(b,c),$$

where

$$\phi(xy) = \iint \theta(xy)\,dx\,dy = \iiiint \mathrm{P}\,dx^2\,dy^2.$$

Dealing in the same way with the three remaining integrals

$$-\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \theta(x_1y_2)\,d\mathrm{X}\,d\mathrm{Y}, \quad -\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \theta(x_2y_1)\,d\mathrm{X}\,d\mathrm{Y}, \quad \text{and} \quad \int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \theta(x_2y_2)\,d\mathrm{X}\,d\mathrm{Y},$$

we find that they become

$$\int_{-c}^{0}\int_{b}^{0}\theta(x_1y_2)\,dx_1\,dy_2, \qquad \int_{c}^{0}\int_{-b}^{0}\theta(x_2y_1)\,dx_2\,dy_1 \qquad \text{and} \qquad \int_{-c}^{0}\int_{-b}^{0}\theta(x_2y_2)\,dx_2\,dy_2$$

respectively, which are equal to

$$\phi(0,0)-\phi(0,-c)-\phi(b,0)+\phi(b,-c),$$

$$\phi(0,0)-\phi(0,c)-\phi(-b,0)+\phi(-b,c),$$

and

$$\phi(0,0)-\phi(0,-c)-\phi(-b,0)+\phi(-b,-c)$$

respectively, where $\phi(xy)$ has the same meaning as before.

Hence, if

$$\phi(xy)=\iiiint \mathrm{P}\,dx^2\,dy^2$$

$$\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\int_{-\frac{1}{2}c-\mathrm{Y}}^{\frac{1}{2}c-\mathrm{Y}}\int_{-\frac{1}{2}b-\mathrm{X}}^{\frac{1}{2}b-\mathrm{X}} \mathrm{P}\,dx\,dy\,d\mathrm{X}\,d\mathrm{Y}$$

is equal to

$$4\phi(0,0)+\Sigma\phi(\pm b\pm c)-2\phi(0,c)-2\phi(0,-c)-2\phi(b,0)-2\phi(-b,0).$$

This expression, obtained from the function $\phi(xy)$ by substituting in it b, c, $-b$, $-c$, 0, 0 in the way indicated, will, in what follows, be designated by

$$\Sigma\phi.$$

2. As an illustration of the above I will indicate the process as applied to the simplest term in P involving $\log(x^2+y^2)$.

Thus to obtain

$$\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\int_{-\frac{1}{2}c-\mathrm{Y}}^{\frac{1}{2}c-\mathrm{Y}}\int_{-\frac{1}{2}b-\mathrm{X}}^{\frac{1}{2}b-\mathrm{X}} \log(x^2+y^2)\,dx\,dy\,d\mathrm{X}\,d\mathrm{Y},$$

we find by simple integrations that

$$\phi(xy)=\iiiint \log(x^2+y^2)\,dx^2\,dy^2=\left(\frac{x^2y^2}{4}-\frac{x^4+y^4}{24}\right)\log(x^2+y^2)$$
$$+\frac{1}{3}\left(x^3y\tan^{-1}\frac{y}{x}+xy^3\tan^{-1}\frac{x}{y}\right)-\frac{25}{24}x^2y^2.$$

By inspection it is seen that

$$\phi(00)=0,$$

$$\Sigma\phi(\pm b,\pm c)=\left(b^2c^2-\frac{b^4+c^4}{6}\right)\log(b^2+c^2)-\frac{25}{6}b^2c^2+\frac{4}{3}\left(b^3c\tan^{-1}\frac{c}{b}+bc^3\tan^{-1}\frac{b}{c}\right),$$

$$\phi(b,0)=\phi(-b,0)=-\frac{b^4}{24}\log b^2,$$

$$\phi(0,c)=\phi(0,-c)=-\frac{c^4}{24}\log c^2.$$

Hence the definite integral above is equal to

$$b^2c^2\left[\log(b^2+c^2)-\frac{b^2}{6c^2}\log\frac{b^2+c^2}{b^2}-\frac{c^2}{6b^2}\log\frac{b^2+c^2}{c^2}-\frac{25}{6}+\frac{4}{3}\left(\frac{b}{c}\tan^{-1}\frac{c}{b}+\frac{c}{b}\tan^{-1}\frac{b}{c}\right)\right].$$

The above is the well-known definite integral used for determining the G.M.D. of a rectangle from itself.

3. To determine

$$\iiiint \mathrm{YQ}\, dx\, dy\, d\mathrm{X}\, d\mathrm{Y},$$

between the given limits.

If

$$\theta(xy) = \iint \mathrm{Q}\, dx\, dy,$$

the result of the integrations with respect to x and y will now be

$$\mathrm{Y}\left\{\theta(x_1y_1)-\theta(x_1y_2)-\theta(x_2y_1)+\theta(x_2y_2)\right\}$$

where x_1, y_1, x_2, y_2 have the same significations as before.

We have now to evaluate four integrals of the type

$$\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \mathrm{Y}\theta(x_1y_1)\, d\mathrm{X}\, d\mathrm{Y}.$$

Proceeding as in § 1, these, affected by their proper signs, become

$$\int_c^0\int_b^0 (\tfrac{1}{2}c-y_1)\,\theta(x_1y_1)\,dx_1\,dy_1-\int_{-c}^0\int_b^0(\tfrac{1}{2}c+y_2)\,\theta(x_1y_2)\,dx_1\,dy_2$$
$$+\int_c^0\int_{-b}^0(\tfrac{1}{2}c-y_1)\,\theta(x_2y_1)\,dx_2\,dy_1-\int_{-c}^0\int_{-b}^0(\tfrac{1}{2}c+y_2)\,\theta(x_2y_2)\,dx_2\,dy_2,$$

so that, if in this case

$$\phi(xy) = \iiiint \mathrm{Q}\, dx^2\, dy^2,$$

$$\phi'(xy) = \int y\, dy \iiint \mathrm{Q}\, dx^2\, dy,$$

and

$$\Delta\phi = \phi(b,c)+\phi(-b,c)-\phi(b,-c)$$
$$-\phi(-b,-c)+2\phi(0,-c)-2\phi(0,c),$$

then

$$\iiiint \mathrm{YQ}\, dx\, dy\, d\mathrm{X}\, d\mathrm{Y}$$

between the given limits is equal to

$$\frac{c}{2}\Delta\phi-\Sigma\phi'$$

where Σ has the signification given to it in § 1.

4. In a similar way it can be shown that

$$\iiiint Y^2 R \, dx \, dy \, dX \, dY$$

between the given limits is equal to

$$\left(\frac{c}{2}\right)^2 \Sigma\phi - 2\left(\frac{c}{2}\right)\Delta\phi' + \Sigma\phi''$$

where, in this case,

$$\phi(xy) = \iiiint R \, dx^2 \, dy^2,$$

$$\phi'(xy) = \int y \, dy \iiint R \, dx^2 \, dy,$$

$$\phi''(xy) = \int y^2 \, dy \iiint R \, dx^2 \, dy,$$

and that

$$\iiiint Y^3 S \, dx \, dy \, dX \, dY$$

between the given limits is equal to

$$\left(\frac{c}{2}\right)^3 \Delta\phi - 3\left(\frac{c}{2}\right)^2 \Sigma\phi' + 3\left(\frac{c}{2}\right)\Delta\phi'' - \Sigma\phi''',$$

where, in this case,

$$\phi(xy) = \iiiint S \, dx^2 \, dy^2,$$

$$\phi'(xy) = \int y \, dy \iiint S \, dx^2 \, dy,$$

$$\phi''(xy) = \int y^2 \, dy \iiint S \, dx^2 \, dy,$$

$$\phi'''(xy) = \int y^3 \, dy \iiint S \, dx^2 \, dy.$$

The result of integration can now be easily written out for integrations involving higher powers of Y.

5. Before proceeding with the integrations it is advisable to have prepared beforehand a table giving

$$\int x^n \log(x^2+y^2) \, dx$$

and

$$\int x^n \tan^{-1}\frac{y}{x} \, dx$$

from $n = 0$ to $n = 7$.

If this be done, and the method indicated above followed, the work presents little difficulty and is not very tedious.

APPENDIX II.

(*Added October* 1, 1913.)

Since writing the above I have determined the sixth order term of the series for L.

In order to do this it was necessary to extend to the sixth order MAXWELL'S series formula (see § 1) for M, the mutual inductance of two unequal coaxal circles which ROSA and COHEN* had already extended to the fifth order.

Thus

$$M = 4\pi a\left[\log\frac{8a}{r}\left\{1+\frac{y}{2a}+\frac{3x^2+y^2}{2^4\,.\,a^2}-\frac{3x^2y+y^3}{2^5\,.\,a^3}\right.\right.$$
$$-\frac{15x^4-42x^2y^2-17y^4}{2^{10}\,.\,a^4}+\frac{45x^4y-30x^2y^3-19y^5}{2^{11}\,.\,a^5}$$
$$\left.+\frac{35x^6-345x^4y^2+45x^2y^4+89y^6}{2^{14}\,.\,a^6}\right\}$$
$$-2-\frac{y}{2\,.\,a}-\frac{x^2-3y^2}{2^4\,.\,a^2}+\frac{6x^2y-y^3}{2^4\,.\,3\,.\,a^3}$$
$$+\frac{93x^4-534x^2y^2-19y^4}{2^{11}\,.\,3\,.\,a^4}-\frac{1845x^4y-3030x^2y^3-379y^5}{2^{12}\,.\,3\,.\,5\,.\,a^5}$$
$$\left.-\frac{1235x^6-17445x^4y^2+12045x^2y^4-7371y^6}{2^{15}\,.\,3\,.\,5\,.\,a^6}\right].$$

When in M we substitute, as explained in § 1, $a+Y$ for a, the term of the sixth order in the variables x, y, and Y becomes equal to U, where

$$U = p+qY+rY^2+sY^3+tY^4+uY^5+vY^6,$$

in which

$$p = 4\pi a\left\{\frac{35x^6-345x^4y^2+45x^2y^4+89y^6}{2^{14}\,.\,a^6}\log\frac{8a}{r}\right.$$
$$\left.-\frac{1235x^6-17445x^4y^2+12045x^2y^4-7371y^6}{2^{15}\,.\,3\,.\,5\,.\,a^6}\right\},$$

$$q = 4\pi a\left\{\frac{-45x^4y+30x^2y^3+19y^5}{2^9\,.\,a^6}\log\frac{8a}{r}+\frac{4635x^4y-6510x^2y^3-1043y^5}{2^{11}\,.\,3\,.\,5\,.\,a^6}\right\},$$

$$r = 4\pi a\left\{\frac{-45x^4+126x^2y^2+51y^4}{2^9\,.\,a^6}\log\frac{8a}{r}+\frac{291x^4-1362x^2y^2-157y^4}{2^{11}\,.\,a^6}\right\},$$

$$s = 4\pi a\left\{\frac{3x^2y+y^3}{2^3\,.\,a^6}\log\frac{8a}{r}-\frac{87x^2y+5y^3}{2^5\,.\,3\,.\,a^6}\right\},$$

$$t = 4\pi a\left\{\frac{3x^2+y^2}{2^4\,.\,a^6}\log\frac{8a}{r}-\frac{87x^2-11y^2}{2^6\,.\,3\,.\,a^6}\right\},$$

$$u = 4\pi a\,.\,\frac{y}{2\,.\,5\,.\,a^6},$$

$$v = 4\pi a\,.\,\frac{1}{2\,.\,3\,.\,5\,.\,a^6}.$$

* 'Bull. Bureau Standards,' 2, p. 364, 1906.

The term of the sixth order in the series for L is the value of the integral

$$\frac{1}{b^2c^2}\iiiint \mathrm{U}\,dx\,dy\,d\mathrm{X}.\,d\mathrm{Y},$$

between the specified limits, and I have found it to be equal to,

$$\frac{4\pi}{2^{16}.\,3\,.\,5\,.\,7\,.\,a^5}\Bigg[(525b^6-1610b^4c^2+770b^2c^4+103c^6)\log\frac{8a}{d}$$
$$+\left(\frac{3633}{10}\,b^6-3220b^4c^2+2240b^2c^4\right)u,$$
$$-\frac{359}{30}\,c^6v-2^{11}\left(\frac{5}{3}\,b^6-4b^4c^2+\frac{7}{5}\,b^2c^4\right)w,$$
$$+\frac{2161453}{2^3.\,3\,.\,5\,.\,7}\,b^6-\frac{617423}{2^2.\,3^2.\,5}\,b^4c^2-\frac{8329}{2^2.\,3\,.\,5}\,b^2c^4+\frac{4308631}{2^3.\,3\,.\,5\,.\,7}\,c^6\Bigg],$$

in which u, v, w, and d have the significations assigned to them in § 2.

The method of integration indicated in this paper renders the determination of L in series form comparatively easy for the special cases of a solenoid ($c/b = 0$), and a flat circular ring coil ($b/c = 0$), uniform current density being assumed.

Thus COFFIN's formula* for a solenoid can be easily obtained, and RAYLEIGH† and NIVEN's formula for a coil whose axial dimension (b) is zero can be extended to the sixth order, giving

$$\mathrm{L} = 4\pi n^2 a\Bigg[\left(1+\frac{c^2}{2^5.\,3\,.\,a^2}+\frac{11c^4}{2^{10}.\,3^2.\,5\,.\,a^4}+\frac{103c^6}{2^{16}.\,3\,.\,5\,.\,7\,.\,a^6}\right)\log\frac{8a}{c}$$
$$-\tfrac{1}{2}+\frac{43c^2}{2^7.\,3^2.\,a^2}+\frac{c^4}{2^5.\,3\,.\,5^2.\,a^4}+\frac{4298579c^6}{2^{19}.\,3^2.\,5^2.\,7^2.\,a^6}\Bigg],$$

which can also be obtained by putting $b = 0$ in the general formula obtained above for L, and remembering that when $b/c = 0$, $v = 1$, and $w' = 1$.

* 'Bull. Bureau Standards,' 2, p. 113, 1906.

† RAYLEIGH, 'Collected Papers,' 2, p. 15.

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